Stretching of vortex lines and generation of vorticity in the three-dimensional complex Ginzburg-Landau Equation

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(February 4, 2008)

The dynamics of curved vortex filaments is studied analytically and numerically in the framework of a three-dimensional complex Ginzburg-Landau equation (CGLE). It is proved that a straight vortex line is unstable with respect to spontaneous stretching and bending in a certain range of parameters of the CGLE, resulting in formation of persistent entangled vortex configurations. The analysis shows that the standard approach relating the velocity of the filament with the local curvature, is insufficient to describe the instability and self-generation of vorticity.

PACS: 74.60.Ge,68.10.-m,05.60.+w

The complex Ginzburg-Landau equation (CGLE), derived some 20 years ago by Newell [1] and Kuramoto [2] has become a paradigm model for a qualitative description of weakly-nonlinear oscillatory media (see for review [3]). Under appropriate scaling of the physical variables, the equation assumes the universal form

$$\partial_t A = A - (1+ic)|A|^2 A + (1+ib)\Delta A, \tag{1}$$

where A is the complex amplitude, b and c are real parameters, and $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is a three-dimensional Laplace operator. Although the equation is formally valid only at the threshold of a supercritical Hopf bifurcation, it has been found that the CGLE often reproduces qualitatively correct phenomenology over a much wider range of the parameters. As a results, the predictions drawn from the analysis of the CGLE (mostly in one and two spatial dimensions, see e.g. [4–6]) were recently successfully confirmed by experiments in optical and chemical systems [7,8]. Moreover, some results obtained from the CGLE (for example, symmetry breaking of spiral pairs) was instructive for interpretation of experiments in far more complicated systems of chemical waves [9] and colonies of amoebae [10,11].

Recently, the dynamics of *three-dimensional* (3D) vortex filaments has attracted substantial attention [12,14–17]. In the context of the 3D CGLE, Gabbay Ott and Guzdar [14] applied a generalization of Keener's method for a scroll vortex in reaction-diffusion systems [18]. They derived that a vortex ring of radius R collapses in finite time over the entire range of parameters (b, c) of Eq. (1) according to the following evolution law

$$\frac{dR}{dt} = -\frac{1+b^2}{R}. \tag{2}$$

This result generalizes Keener's ansatz by including the curvature-induced shift of the filament's wavenumber and corrects the erroneous estimate of Ref. [12]. Thereby, as follows from Eq. (2), vorticity initially existing in the system will always decay (if, of course, there is no bulk instability of the waves emitted by the vortex filament), and under no condition can the vortex filament expand. The persistence of vortex filaments, known under some conditions for reaction-diffusion systems, was attributed in the context of the CGLE to the effect of higher-order corrections to the CGLE.

In this Letter we show that under very general conditions and in an extensive part of the parameter space vortex filaments *expand* and result in persistent vortex configurations even if there is no bulk instability of emitted waves. The condition for the expansion of the vortex filament is simply $b > b_c(c) \gg 1$. In this limit Eq. (2) is not valid, because formally higher-order corrections, omitted in Eq. (2), cause severe instability of the filament and persistent stretching. This instability is a three-dimensional manifestation of the two-dimensional core instability of spiral waves (called in Ref. [5] *acceleration instability*). It originates from breakdown of the Galilean invariance of the CGLE for any $\epsilon = 1/b \neq 0$, causing spontaneous acceleration of the spiral waves for $\epsilon \leq \epsilon_c$ [5]. Although in 2D the instability is relatively weak (the growth rate is proportional to ϵ), the situation is different in three dimensions. As we will show, the bending of the filament greatly enhances the instability, resulting in formation, after some transient, of a highly entangled and dense vortex configuration. The condition $b \gg 1$ is readily fullfiled for many physical and chemical systems. For example, in the context of nonlinear optics, where the CGLE can be derived from the Maxwell-Bloch equation in the good cavity limit [13], this parameter is very small: $\epsilon = 1/b \sim 10^{-4} - 10^{-3}$. For an oscillating chemical reaction the diffusion rates of various components can be varied over a wide range by adding extra chemicals.

As a test for instability, we consider the dynamics of a

weakly curved vortex filament. We apply the generalization of the method of Ref. [5] for the case of 3D vortices, and make perturbations near the 2D spiral wave solution to the CGLE. For convenience, we redefine $\mathbf{r} \to \mathbf{r}/\sqrt{b}$. Then Eq. (1) assumes the form

$$\partial_t A = A - (1+ic)|A|^2 A + (\epsilon + i)\Delta A. \quad (3)$$

In the following discussion, we assume $0 < \epsilon \ll 1$ to be a small parameter. Our objective is to relate the acceleration of the vortex filament $\partial_t \mathbf{v}$ with the velocity $\mathbf{v}$ and local curvature κ of the filament. We obtain that for $\epsilon \ll 1$ the equation of filament motion can be written as

$$\partial_t \mathbf{v} + \hat{K}[\epsilon \mathbf{v} + (1+\epsilon^2)\kappa \mathbf{N}] = 0, \quad (4)$$

where $\mathbf{N}$ is the unit vector pointing toward the center of curvature, and $\mathbf{B}$ is the unit vector perpendicular to the filament and $\mathbf{N}$. Velocity and acceleration have correspondingly two components, $\mathbf{v} = (v_N, v_B), \partial_t \mathbf{v} = (\partial_t v_N, \partial_t v_B)$. $\hat{K}$ is the friction tensor satisfying $K_{11} = K_{22}, K_{12} = -K_{21}$. Dropping the first term in Eq.(4), we reproduce the result of Ref. [14] for the collapse rate $v_N = -(1+\epsilon^2)\kappa/\epsilon$ (because $\kappa = 1/R$). However, if $K_{11} < 0$, which we will show is the case for $\epsilon < \epsilon_c(c)$, the acceleration has a significant effect and leads to an ultimate expansion of the vortex filaments and the persistence of entangled vortex configurations. In principle, the numerical value of $\epsilon_c(c)$ does not need to be very small. For example, for $c = 0.5$ we obtained $\epsilon_c \approx 0.165$, which corresponds to $b \approx 6$.

In order to develop the perturbation theory for a weakly-curved vortex filament in 3D, we begin with the stationary one-armed isolated spiral solution to Eqs. (1,3), which is of the form

$$A_0(r,\theta) = F(r) \exp i[\omega t \pm \theta + \psi(r)] \,, \quad (5)$$

where (r,θ) are polar coordinates. The real functions F and ψ have the following asymptotic behavior $F(r) \to \sqrt{1-\epsilon k^2}$, $\psi'(r) \to k$ for $r \to \infty$ and $F(r) \sim r$, $\psi'(r) \sim r$ for $r \to 0$. The wavenumber k of the waves emitted by the spiral is determined uniquely for given ε, c [19]. For $\epsilon = 0$ one has a type of Galilean invariance and then, in addition to the stationary spiral, there exists a family of spirals moving with arbitrary constant velocity $\mathbf{v}$ [5],

$$A(r,t) = F(r') \exp i[\omega' t + \theta + \psi(r') - \frac{\mathbf{r}' \cdot \mathbf{v}}{2}], \quad (6)$$

where $\mathbf{r}' = \mathbf{r} + \mathbf{v}t$, $\omega' = \omega + \mathbf{v}^2/4$, and the functions F, ψ are those of Eq. (5). (This invariance holds for any stationary solution). For $\epsilon \neq 0$ the diffusion term $\sim \varepsilon \Delta A$ destroys the family and leads to acceleration/deceleration of the spiral proportional to ϵv, depending on the value of ϵ. For $\epsilon < \epsilon_c(c)$ the spiral is unstable with respect to spontaneous acceleration since $K_{11} < 0$ [5].

Let us now consider the dynamics of an almost straight vortex line in the filament-based coordinate system (see for details [14]). The position in space X is represented by local coordinates $s, \tilde{x}, \tilde{y}$, where s is the arclength along the filament, and $\mathbf{X} = \mathbf{R}(s) + \tilde{x}\mathbf{N}(s) + \tilde{y}\mathbf{B}(s)$, where $\mathbf{R}$ is the coordinate of the filament. In this basis the weakly curved filament moving with velocity $\mathbf{v}$ can be written in the form

$$A(r,t) = F(r') \exp\left[i \left(\omega' t + \theta + \psi(r') - \frac{\mathbf{r}' \cdot \mathbf{v}}{2} + \frac{\delta k_x \tilde{x}}{2} \right) \right] + W(r', \theta, s). \quad (7)$$

Here $\mathbf{r}' = \mathbf{r} + \mathbf{vt}$, W is the perturbation to the spiral solution which we require to be small, and δk_x is the correction to the wavenumber due to curvature of the filament. In principle, this correction can be absorbed in W but it is convenient to retain this form since we can cancel part of the perturbation exactly by adjusting δk_x [14]. The perturbation procedure to derive Eq. (4) is practically identical to that of Refs. [4,5]. Substituting the ansatz Eq. (7) into the CGLE, and assuming $|\partial_t v|, \kappa \ll 1$, one obtains from Eq. (3) in first order in ϵ an inhomogeneous linear equation for the correction W, which is of the form $\hat{L}W = H$, where $\hat{L}W = (1+i\omega - 2(1+ic)|A_0|^2)W - (1+ic)A_0^2 W^* + i\Delta_\perp W$, $\Delta_\perp = \partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2$. In first order the W-independent part H has two terms, $H = H_{tr} + H_{rot}$, where H_{tr} contributes to the displacement of the vortex filament, and H_{rot} contributes to the shift of the vortex frequency. We are interesting only in H_{tr} (the contribution H_{rot} is not responsible for the instability), which is of the form:

$$H_{tr} = -\mathbf{r}' \partial_\mathbf{t} \mathbf{v} \frac{iA_0}{2} + i\epsilon \mathbf{v} \nabla A_0 + (\epsilon + i)(\kappa - i\delta k_x)\nabla A_0. \quad (8)$$

The derived system of equations is very close to that considered in the Ref. [5] in the context of the acceleration instability of a spiral wave in 2D. The acceleration can be obtained as a result of the solvability condition requiring that W be regular at the core of the spiral and does not diverge *exponentially* at large r. (Slower, power-law, divergence of W is permitted). To impose the solvability condition a specific numerical procedure is required (see for details [4,5]). Eq. (8) contains an extra term $(\epsilon + i)(\kappa - i\delta k_x)\partial_{\tilde{x}} A_0$ with respect to that considered in [5], which originates from the Laplace operator in a local basis (in the limit of small curvature κ and torsion τ): $\Delta = -\kappa \partial_{\tilde{x}} + \partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2 + \partial_s^2 + \ldots$. However, additional numerical integration is not necessary to account for the effect of this term, because, as was found in Ref. [14], this term can be canceled *exactly* to first order in κ by a proper choice of $\delta k_x = -\epsilon\kappa$. After this modification, H_{tr} assumes the form $H_{tr} = -\mathbf{r}' \partial_\mathbf{t} \mathbf{v} \frac{iA_0}{2} + i\epsilon \bar{\mathbf{v}} \nabla A_0$, where $\bar{v}_N = v_N + (1+\epsilon^2)\kappa/\epsilon$ and $\bar{v}_B = v_B$. It is easy to see that

it now coincides with the corresponding function considered in Ref. [5]. Following the lines of analysis of Ref. [5], we observe that the relation between $\partial_t v$ and $\bar{v}$ is identical to that considered in [5], and is of the form

$$\partial_{\mathbf{t}} \mathbf{v} + \epsilon \hat{K} \bar{\mathbf{v}} = 0. \tag{9}$$

Thus we obtain Eq. (4). The coefficients of the friction tensor K_{ij} coincide with those calculated in Ref. [5] using numerical matching, and for $\epsilon < \epsilon_c$ it was found that $K_{11} < 0$, which would guarantee the instability with respect to spontaneous acceleration of the vortex filament. Moreover, since in 3D the direction of motion of the filament in general varies along the filament, this instability results in stretching of the vortex.

In the case of the acceleration instability, $K_{11} < 0$ and the curvature has a strong destabilizing effect. Indeed, for an almost straight vortex parallel to the axis z, we can parameterize the position along the filament by the z coordiante: $(X_0(z), Y_0(z))$. Since in this limit the arclength s is close to z, the curvature correction to the velocity $\kappa\mathbf{N}$ is simply $\kappa\mathbf{N} = -(\partial_z^2 X_0, \partial_z^2 Y_0)$. After simple algebra one derives the following relation for the growth rate $\lambda(k)$ of linear perturbation $X_0(z), Y_0(z) \sim \exp[\lambda(k)t + ikz]$: $\lambda^2 + (K_{11} + iK_{12})(\epsilon\lambda + k^2) = 0$. For $k \gg \epsilon$ we obtain $\lambda \approx \sqrt{-(K_{11} + iK_{12})}k \gg \epsilon$. Thus, for finite k the growth rate $\lambda(k)$ may significantly exceed the increment of the acceleration instability in 2D (corresponding to $k = 0$): $\lambda_0 = -\epsilon(K_{11} + iK_{12})$. We can expect that, as a result of such an instability, highly-curved vortex filaments will be formed. Therefore, the above considered "small-curvature" approximation can be valid only for finite time. Moreover, it is naive to expect a saturation of this instability in a steady-state configuration with finite curvature. In contrast, we suggest that frequently reconnection of various parts of the filaments, formation of vortex rings etc, will result in persistent spatio-temporal dynamics of a highly-entangled vortex state.

In order to follow further development of the instability we performed numerical simulations of the 3D CGLE. We studied a system of 50^3 dimensionless units of Eq. (3) with no-flux boundary conditions. The numerical solution was performed on an R10000 SGI workstation by an implicit Crank-Nicholson algorithm. The number of grid points was 100^3. We performed simulations for $\epsilon = 0.02$ in the parameter regime away from amplitude turbulence in 2D [5]. As an initial condition we used a straight vortex line perturbed by small noise. The results of the simulations are shown in Fig. 1. We see that the length of the vortex line grows. The dynamics seems to be very rapidly varying in time, and the line intersects itself many times forming numerous vortex rings. The long-time dynamics show, however, some kind of saturation when a highly-entangled vortex state is formed and the length of the line cannot grow anymore due to repulsive interaction between closely packed line segments. The dependence of the line length on time is shown in Fig. 2. As a measure of the filament length L we used the following quantity: $L \approx S_0^{-1} \int \Theta(A_0 - |A(x, y, z)|) dxdydz$, where Θ is a step function: $\Theta(x) = 1$ if $x > 0$ and $\Theta = 0$ otherwise. $A_0 = 0.1$ was used as a threshold value to identify the vortex. S_0 is a constant determined from the condition that for the straight line the above integral coincides with the actual length. We can identify two distinct stages of the dynamics: first, fast growth of the length; second, oscillations of the line's length around some mean value. Remarkably, we observed an increase in the amplitude of the oscillation moving though negative c values in the regime of spatio-temporal intermittency in the 2D CGLE. It is plausible that the symmetry breaking mechanism responsible for persistent intermittent behavior in the 2D system still has some importance in 3D.

Persistent entangled vortex configurations are known from numerical simulations of excitable reaction-diffusion systems [15,16]. Recently, elongation of a vortex filament was also related with the meandering instability of a vortex core in 2D [20]. However, there is also a significant difference between these two phenomena. For the reaction-diffusion systems the meandering instability is typically supercritical, and may be saturated at a relatively small amplitude of oscillations. In application to 3D systems, one expects that highly-curved vortex filaments will not develop, and for this reason the intersections are very seldom. This picture is consistent with the numerical simulations. Consequently, one may expect that the analysis performed on the basis of a small-curvature approximation will be valid for a very long time and can capture the actual dynamics of the filament. In the CGLE the 2D instability is subcritical and the velocity of the spiral in an infinite system can grow arbitrarily until additional topological defects nucleate in the wake of the accelerating spiral. In the 3D situation we have shown that curvature of the filament is even a destabilizing factor. Our numerical simulations show that the filament does not approach any steady-state. In contrast, it shows "violent" intermittent behavior, with numerous reconnections and splitting of vortex rings. As a result, the approximate equation of motion (4) must be considered only as a test for instability rather than long-time evolution.

In conclusion, we have derived an equation of motion for the vortex filament in the CGLE. We have found that in a wide range of parameters of the CGLE the vortex filament is unstable with respect to spontaneous stretching, resulting in the formation of persistent entangled vortex configurations. This emphasizes the deficiency of previous approaches relating local filament velocity to local curvature. Our result could be verified in experiments with autocatalytic chemical reactions in gels in the regime of oscillatory instability. Also, the limit of a large cross-diffusion ratio can probably be achieved by doping with additional chemicals, thus changing the rel-

ative mobility of one chemical species with respect to another. Recently, the amplitude equation governing the dynamics of an elastic rod was derived [21]. We note that our instability can be formally interpreted as a dynamics of a thin rod with *negative* elasticity. We also speculate that our results are relevant for inviscid hydrodyanmics. In the limit of $b, c \to \infty$, Eq. (1) reduces to the defocusing nonlinear Schrödinger equation (NSE), which is a paradigm model for compressible inviscid hydrodynamics. Although the vortex lines are stable in the framework of the NSE, the corrections arising from the CGLE cause the destabilization and self-generation of vorticity.

We are grateful to A. Newell, D. Levermore, C. Doering and R. Goldstein for illuminating discussions. This work was supported by the U.S. Department of Energy under contracts W-31-109-ENG-38 (IA) and ERW-E420 (AB). The work of IA was also supported by the NSF, Office of Science and Technology Center under contract No. DMR91-20000.

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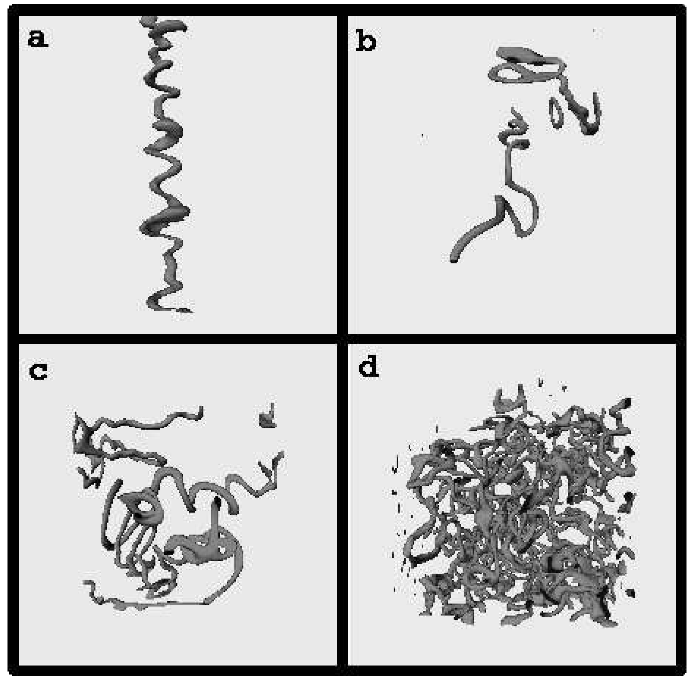

FIG. 1. Instability of a straight vortex filament. 3D isosurfaces of $|A(x, y, z)| = 0.1$ for $\epsilon = 0.02$, $c = -0.03$, shown at four times: 50 (a), 150 (b), 250 (c), 500(d).

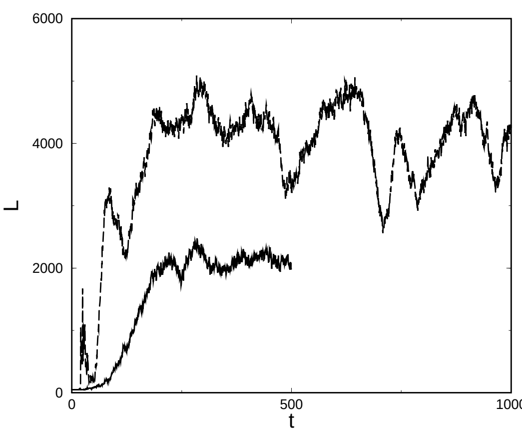

FIG. 2. The dependence of filament length L on time. Solid line corresponds to $\epsilon = 0.02, c = -0.03$; dashed line corresponds to $\epsilon = 0.02, c = -0.5$.